Comment on "A Stress-Strain Approximation in Plasticity"

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In a recent Note, A. D. Fine[1] has shown that the multiaxial stress-strain diagram can be approximated by the uniaxial stress-strain diagram. He has also noted that for a linear strain hardening material (monotonically increasing loading) the diagrams are identical if Poisson's ratio μ is 0.5, and that the approximate error is of the order of 10% for $\mu = 0$.

This writer would like to make the following comments. Although Mr. Fine's paper is a valuable tool for engineers and analysts, the results and conclusions are hardly surprising or new. Several years ago Vol'mir[2] showed that the uniaxial stress-strain diagram can be used to approximate the inelastic behavior of biaxially loaded structural members. He has proved that for an incompressible material ($\mu = 0.5$) the diagrams coincide. His proof is based on an assumption made by Kachanov,[3] namely that the stress-strain expressions can be separated into an elastic and an inelastic part. The final conclusion reached is that the secant moduli are identical in the two diagrams. Vol'mir's conclusions are therefore valid for any arbitrary stress-strain relation and are not restricted to the linear strain hardening case.

The relative error estimate (about 10% when $\mu = 0$) in Ref. 1 is of academic interest only as far as structural materials are concerned. It is a matter of experimental evidence that most common materials are characterized by a Poisson ratio of about 0.5 in the plastic range.

The results and conclusions of Refs. 2 and 3 have been used quite extensively in the Russian literature over the past decade since they are of great practical importance. It is felt by this writer that the results of the aforementioned papers will be of importance in elastic-plastic analyses, since they lend themselves readily to computer applications.

References

[1] Fine, A. D., "A Stress-Strain Approximation in Plasticity," *AIAA Journal*, Vol. 5, No. 10, Oct. 1967, pp. 1896–1897.
[2] Vol'mir, A. S., "Stability of Elastic Systems," FTD-MT-64-335, U.S. Air Force Systems Command.
[3] Kachanov, L. M., *Principles of the Theory of Plasticity* (in Russian), Gostekhizdat, 1956.

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Reply by Author to H. M. Haydl

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The basic motivation for the original Note[1] was to discuss compressibility effects during plastic loading by studying parametrically the variation of Poisson's ratio ν. The author feels that this objective has been achieved and believes furthermore that the analysis involved in this work will prove useful both from a theoretical and a practical point of view. Although many structural materials in the past have exhibited behavior nearly incompressible during plastic loading, the more exotic materials now being used in industry may no longer exhibit such characteristics (see, for example, a recent paper by Chu and Hashin[2]). Consequently, the author feels that the relative error estimate of 10% may prove extremely important in future structural engineering problems.

The author thanks Mr. Haydl for pointing out the work of Vol'mir.[3] The result obtained in the original Note[1] thus can be considered as a generalization of that analysis. Equation (1) of that note clearly reduces to Vol'mir's result, if one sets $\nu = \frac{1}{2}$; furthermore, this expression, i.e., (1) in Ref. 1, is not restricted to the linear hardening case alone as Mr. Haydl has implied, but is applicable in the general loading case for any value of ν in the physically acceptable range.

The author agrees with Mr. Haydl that analyses such as those given in Refs. 1 and 3 represent extremely useful tools for practical engineers. It is hoped, therefore, that the ideas contained therein, as well as those discussed here and in Mr. Haydl's comments, will reach as large an audience as possible, and that these conclusions will be utilized in future engineering problems involving plastic loading phenomena.

References

[1] Fine, A. D., "A Stress-Strain Approximation in Plasticity," *AIAA Journal*, Vol. 5, No. 10, Oct. 1967, pp. 1896–1897.
[2] Chu, T. Y. and Hashin, Z., "Plastic Behavior of Composites and Porous Media Under Hydrostatic Pressure," *Proceedings of the Fifth U.S. National Congress of Applied Mechanics*, June 1966, p. 545.
[3] Vol'mir, A. S., "Stability of Elastic Systems," U.S. Air Force Systems Command, FTD-MT-64-335.

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Comments on "Solution of the Algebraic Matrix Riccati Equation via Newton-Raphsen Iteration"

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In a recent Note, Blackburn[1] presents a method of obtaining the steady-state solution of the matrix Riccati equation using the Newton-Raphsen technique. If the order of the system n is large, however, this method of solution, which requires a matrix inversion of a $\frac{1}{2}n(n + 1)$-dimensional Jacobian at each iteration, is impractical because of the limitations in the computer storage space and the high consumption of the computing time. It is the intent of this Comment to present a modification to the algorithm and its method of solution, so that the aforementioned drawbacks can be circumvented.

Blackburn's iterative scheme for the solution of the algebraic matrix Riccati equation

$$-PA - A'P + PGP - Q = 0$$

is essentially as follows:

$$[P_{k+1}] = [P_k] + \{I \times (A - GP_k)' + (A - GP_k)' \times I\}^{-1} \cdot [-P_kA - A'P_k + P_kGP_k - Q] \quad (1)$$

where the subscript k indicates the kth iteration, and the notations $[C]$ and $\{D\}$ denote, respectively, a $\frac{1}{2}n(n + 1)$-dimensional column vector and a $\frac{1}{2}n(n + 1) \times \frac{1}{2}n(n + 1)$ matrix appropriately formulated from the $n \times n$ matrix

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C and the $n^2 \times n^2$ matrix D. Writing (1) as

$$[P_{k+1}] = [P_k] + \{I \times (A - GP_k)' + (A - GP_k)' \times I\}^{-1} \cdot [-P_k(A - GP_k) - (A - GP_k)'P_k - P_kGP_k - Q] \quad (2)$$

and regrouping terms in (2) gives

$$[P_{k+1}] = \{I \times (A - GP_k)' + (A - GP_k)' \times I\}^{-1} \cdot [-P_kGP_k - Q] + [P_k] - \{I \times (A - GP_k)' + (A - GP_k)' \times I\}^{-1}[P_k(A - GP_k) + (A - GP_k)'P_k]\} \quad (3)$$

With the use of the Kronecker product, it can be shown that

$$[P_k(A - GP_k) + (A - GP_k)'P_k] = \{I \times (A - GP_k)' + (A - GP_k)' \times I\}[P_k] \quad (4)$$

Substituting (4) in (3) yields

$$[P_{k+1}] = \{I \times (A - GP_k)' + (A - GP_k)' \times I\}^{-1} \cdot [-P_kGP_k - Q] \quad (5)$$

In view of (4), (5) can be further simplified to give the desired iterative scheme

$$P_{k+1}(A - GP_k) + (A - GP_k)'P_{k+1} = -P_kGP_k - Q \quad (6)$$

The modified iterative scheme requires at each iteration only to solve a matrix Lyapunov equation of the type

$$SF + F'S = -T \quad (7)$$

An algorithm that requires little computer storage space and is fast for solving (7) is as follows[2]:

$$S_1 = (h/3)(4\Phi'T\Phi + 2T)$$

$$S_{i+1} = (\Phi')^{2i}S_i(\Phi)^{2i} + S_i \qquad i = 1,2,3, \ldots$$

$$S = \lim_{i\to\infty} S_i - \frac{h}{3}T$$

where

$$\Phi = [I - (h/2)F + (h/12)F^2]^{-1}[I + (h/2)F + (h/12)F^2]$$

and h is the discrete time interval. This algorithm requires $4n^2$ words of computer memory and converges to 6 significant figure accuracy typically in $30n^3$ time units.† For example, for a 50th-order system, the computer storage space and the computer time required are $10K$ words and 3.75×10^6 time units per iteration, respectively, whereas the corresponding figures for Blackburn's scheme are $1636K$ words and 6.9×10^8 time units per iteration.‡

It is interesting to note that (6) is identical to the algorithm reported by Puri et al.[3] and Kleinman.[4] However, it appears that their method of solution is based on (5) and consequently suffers the same limitations as (1).

† One time unit is equivalent to the time (in microseconds) required to perform one multiplication and one addition in a digital computer.

References

[1] Blackburn, T. R., "Solution of the Algebraic Matrix Riccati Equation via Newton-Raphsen Iteration," *AIAA Journal*, Vol. 6, No. 5, May 1968, pp. 951–953.

[2] Davison, E. J. and Man, F. T., "On the Numerical Solution of $A'Q + QA = -C$," *IEEE Transactions on Automatic Control*, Vol. 13, No. 4, Aug. 1968.

[3] Puri, N. N. and Gruver, W. A., "Optimal Control Design via Successive Approximations," *8th Annual Joint Automatic Control Conference*, AIAA, June 1967, pp. 335–344.

[4] Kleinman, D. L., "On an Iterative Technique for Riccati Equation Computations," *IEEE Transactions on Automatic Control*, Vol. AC-13, No. 1, Feb. 1968, pp. 114–115.

‡ Time estimation is based on the fact that Gaussian elimination is a $\frac{1}{3}r^3$ time unit process, where r is the number of equations to be solved.

Errata: "Thermally Induced Membrane Stress in a Circular Elastic Shell"

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In the third line of Eq. (15), move the closing bracket to the right, to follow $(\tau - x)$. In the last line of that equation, $\tau - x$ should be enclosed in parentheses. In the second line of Eq. (18), the 8 should be changed to 2 in the numerator of the fraction following the summation sign.

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Announcement: 1968 Author and Subject Indexes

It has been the custom to publish the annual author and subject indexes of the AIAA journals in the last issue of the year. This year, however, with the approval of the Publications Committee, we will publish a combined index of the four journals (*AIAA Journal, Journal of Spacecraft and Rockets, Journal of Aircraft, and Journal of Hydronautics*). All topic headings will be included, whether or not anything on that subject was published. The index will be mailed to all subscribers to the journals in January 1969. We hope that readers will find the combined index more convenient to use than four separate ones.

Ruth F. Bryans
Director, Scientific Publications